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THERMOELASTIC STRESSES IN A HALF-SPACE HEATED BY A CONCENTRATED ENERGY FLUX*

L.N. GERMANOVICH, I.D. KILL and N.S. TSODOKOVA

An exact solution is obtained for the problem and also a simple approximate solution convenient for computations for small times (its error is estimated) that is valid for any absorption coefficients. In the special case of a zero absorption coefficient, the solution is simplified and can be written in elementary functions (an example is presented). In this case new qualitative features of the stress field are found that are not inherent in other methods of heating the half-space. For fairly large absorption coefficients (a criterion is given), a still more simple and convenient closed solution for computations is successfully obtained which can also be expressed in terms of elementary functions (an example is presented). In the case of both large and small absorption coefficients the stress field is analysed and its isolines are constructed.

In a number of cases, temperature stresses that can be the cause of brittle fracture /1-5/ can occur in a solid subjected to a constant energy flux (a laser beam, an electron beam, etc.). The temperature stresses in the body under exposure are studied below on the basis of the extensively utilized model of an elastic half-space (/2-7/, say). It is assumed that internal distributed heat sources whose density decreases exponentially with depth (Bouger's law /5, 8/) act in the half-space. Convective heat transfer from a zero-temperature medium occurs on the half-space boundary. This model is quite adequate and allows a determination of the thermoelastic stresses at both great depths and at depths of the order of the characteristic absorption scale or less.

The plane thermoelasticity problem for a half-space with heat sources was solved in /9/. However, real high-energy beams ordinarily possess axial symmetry. The temperature and thermoelastic stresses in the half-space in /3/ were found by numerical integration of two improper integrals, in the form of which the exact solution is represented, for the case of a uniform energy distribution over the transverse section of a cylindrical beam. An attempt to construct

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the approximate solution for the case of an axisymmetric Gaussian energy distribution in the transverse beam section, high absorption coefficients and short heating times was undertaken in /7/ where the solution is constructed in the form of a series in a system of hypergeometric functions. The solution of the problem for the general case is of interest since the real energy distribution may be far from uniform and Gaussian /10, 11/ while the absorption coefficient is often not large (for transparent materials). Moreover, it is quite desirable to have such a simple approximate solution whose error is controlled.

1. We consider an elastic half-space $z \geqslant 0$ in the cylindrical coordinates r, φ, z on whose boundary heat transfer occurs according to Newton's law with the medium $z < 0$ that has zero temperature. From the time $t = 0$ distributed heat sources with density $q = q_0 f(r) e^{-\gamma z}$ act in the half-space, where the function $f(r)$ allows of a Hankel transformation and the domain of its values is the segment $[0, 1]$. It is required to find the temperature field and the stress field within the half-space whose initial temperature is $T = 0$.

Changing to dimensionless quantities

$$T' = \frac{Tk}{q_0\delta^2}, \quad r' = \frac{r}{\delta}, \quad z' = \frac{z}{\delta}, \quad h' = h\delta, \quad \gamma' = \gamma\delta, \quad t' = \frac{at}{\delta^2} \tag{1.1}$$

where a is the thermal diffusivity, k is the thermal conductivity, h is the relative heat transfer coefficient, and δ is an arbitrary linear dimension, and omitting the primes in writing the dimensionless quantities, for brevity, we obtain a heat conduction boundary value problem

$$\frac{\partial T}{\partial t} = \Delta T + f(r) e^{-\gamma z}; \quad T|_{t=0} = 0, \quad \frac{\partial T}{\partial z}\Big|_{z=0} = hT|_{z=0} \tag{1.2}$$
$$|T(r, z, t)| < \infty \quad \left(\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)$$

Applying a Laplace transformation in t and a Hankel transformation in r to (1.2), we obtain a boundary value problem which we solve and invert the Hankel transform to find

$$T^*(r, z, s) = -\int_0^\infty F(z, \lambda, s)\, \lambda f^H(\lambda)\, J_0(\lambda r)\, d\lambda \tag{1.3}$$
$$F(z, \lambda, s) = \frac{(h + \gamma) e^{-\omega z}}{s(\omega^2 - \gamma^2)(h + \omega)} - \frac{e^{-\gamma z}}{s(\omega^2 - \gamma^2)}, \quad \omega = \sqrt{s + \lambda^2}$$
$$T^* = L_s[T] = \int_0^\infty T(r, z, t) e^{-st}\, dt, \quad f^H(\lambda) = \int_0^\infty r f(r)\, J_0(\lambda r)\, dr$$

where J_0 is a Bessel function of the first kind.

We find the stress transforms by the method of thermoelastic dispalcement potential (/12/, p.21). We will first examine certain general problems. Let the heat conduction equation have the form $\partial T/\partial t = a\Delta T + Q(M, t)$, where M is understood to be a point of the body being heated (a set of coordinates), and dimensional quantities are used for generality. The well-known representation of the thermoelastic potential (/13/, p.484) does not permit direct utilization of the very convenient Parkus method (/12/, p.43). However, a combination of ideas (/12/, p. 43 and /13/, p.484) is possible. Following (/13/, p.484) we represent the thermoelastic potential in the form

$$\Phi = m\int_0^t (aT + \Psi)\, dt + \Phi_0 + \Phi_1 t, \quad m = \alpha\frac{1 + \nu}{1 - \nu} \tag{1.4}$$

where the functions $\Psi(M, t)$, $\Phi_0(M)$ and $\Phi_1(M)$ satisfy, respectively, the equations $\Delta\Psi = Q$, $\Delta\Phi_0 = mT_0$, $\Delta\Phi_1 = 0$, T_0 is the temperature in the half-space at the initial time $t = 0, \nu$ is Poisson's ratio, and α is the coefficient of linear expansion. By comparison with the representation in /13/, the component $\Phi_1 t$ is added to (1.4). It can be shown in the same way as was done earlier (/12/, pp.26 and 27) that this is allowable. Now applying a Laplace transform to (1.4) and using well-known reasoning (/12/, p.43), we find the following representation of the thermoelastic potential in the general case

$$\Phi^*(M, s) = \{m[a(sT^* - T_\infty) + s\Psi^* - \Psi_\infty] + s\Phi_0\} s^{-2} \tag{1.5}$$
$$T_\infty = \lim_{s\to 0} sT^*, \quad \Psi_\infty = \lim_{s\to 0} s\Psi^*$$

Returning to the case under consideration, we can set $\Phi_0 = 0$ (since $T_0 = 0$). Now since Ψ is independent of the time, we have $\Psi^* = \Psi/s$, $\Psi_\infty = \Psi$. Hence and from (1.5) it follows that the representation of the Parkus transform of the thermoelastic potential holds in the special case of the internal heat source density independent of the time (and zero initial temperature). Using it (i.e., (1.5) for $s\Psi^* = \Psi_\infty$, $\Phi_0 = 0$), we obtain

$$\Phi^*(r, z, s) = \frac{m(sT^* - T_\infty)}{s^2}, \quad T_\infty = \lim_{s\to 0} sT^* \tag{1.6}$$

Again returning to the dimensionless quantities (1.1) and substituting (1.3) into (1.6), we find

$$\Phi^* = -m\int_0^\infty \Big\{\frac{(h+\gamma)e^{-\omega z}}{s^2(\omega^2-\gamma^2)(h+\omega)} + \frac{e^{-\gamma z}}{s(\lambda^2-\gamma^2)(\omega^2-\gamma^2)} + \tag{1.7}$$

$$\frac{(h+\gamma)e^{-\lambda z}}{s^2(\lambda^2-\gamma^2)(h+\lambda)}\Big\}\lambda f^H(\lambda) J_0(\lambda r)\, d\lambda$$

We find the transforms of the stresses corresponding to (1.7) by means of (1.32) from /12/. To satisfy the boundary conditions

$$\sigma_{zz}|_{z=0} = \sigma_{rz}|_{z=0} = 0 \tag{1.8}$$

the solution obtained in this manner should be combined with the "temperature-free" solution. It is determined by using the Love biharmonic function

$$L^*(r, z, s) = \int_0^\infty [A(s,\lambda)z + B(s,\lambda)] J_0(\lambda r) e^{-\lambda z}\, d\lambda \tag{1.9}$$

from (8.10) in /14/.

Finally, we have for the transforms of the desired stresses (G is the shear modulus)

$$\frac{\sigma_{rr}^*}{D} = -T^* + \int_0^\infty \Big\{\frac{\lambda F(z,\lambda,s)}{s} + [2h + (1-hz)\lambda - \tag{1.10}$$

$$z\lambda^2]\frac{F(0,\lambda,s)}{s}e^{-\lambda z} + \xi\Big[\frac{1}{s}(\lambda e^{-\gamma z} - (2\gamma - \lambda - \lambda\gamma z +$$

$$\lambda^2 z)e^{-\lambda z}) + \frac{2-\lambda z}{\lambda+\gamma}e^{-\lambda z} + \lambda F_1(z,\lambda)\Big]\Big\}\lambda^2 f^H(\lambda) J_0(\lambda r)\, d\lambda -$$

$$\int_0^\infty \Big\{\frac{\lambda F(z,\lambda,s)}{s} + [2h(1-\nu) + (1-2\nu-hz)\lambda - z\lambda^2]\frac{F(0,\lambda,s)}{s}e^{-\lambda z} +$$

$$\xi\Big[\frac{1}{s}(\lambda e^{-\gamma z} + [2\gamma(\nu-1) + (\gamma z + 1 - 2\nu)\lambda - z\lambda^2]e^{-\lambda z}) +$$

$$\frac{2-2\nu-\lambda z}{\lambda+\gamma}e^{-\lambda z} + \lambda F_1(z,\lambda)\Big]\Big\}\lambda^2 f^H(\lambda)\frac{J_1(\lambda r)}{\lambda r}\, d\lambda$$

$$\frac{\sigma_{\varphi\varphi}^*}{D} = -T^* + 2\nu\int_0^\infty \Big\{\frac{(h+\lambda)F(0,\lambda,s)}{s}e^{-\lambda z} - \xi\Big(\frac{\gamma-\lambda}{s} -$$

$$\frac{1}{\lambda+\gamma}\Big)e^{-\lambda z}\Big\}\lambda^2 f^H(\lambda) J_0(\lambda r)\, d\lambda + \int_0^\infty \Big\{\frac{\lambda F(z,\lambda,s)}{s} + [2h(1-\nu) +$$

$$(1-2\nu-hz)\lambda - z\lambda^2]\frac{F(0,\lambda,s)}{s}e^{-\lambda z} + \xi\Big[\frac{2-2\nu-\lambda z}{\lambda+\gamma}e^{-\lambda z} -$$

$$\frac{1}{s}[2\gamma(1-\nu) - (1-2\nu+\gamma z)\lambda + z\lambda^2]e^{-\lambda z} + \frac{\lambda}{s}e^{-\gamma z} +$$

$$\lambda F_1(z,\lambda)\Big]\Big\}\lambda^2 f^H(\lambda)\frac{J_1(\lambda r)}{\lambda r}\, d\lambda$$

$$\frac{\sigma_{zz}^*}{D} = \int_0^\infty \Big\{-\frac{\lambda F(z,\lambda,s)}{s} + \lambda(1 + hz + \lambda z)\frac{F(0,\lambda,s)}{s}e^{-\lambda z} +$$

$$\xi\Big[\frac{\lambda(1-\gamma z+\lambda z)}{s}e^{-\lambda z} - \frac{\lambda}{s}e^{-\gamma z} + \frac{\lambda z}{\lambda+\gamma}e^{-\lambda z} +$$

$$\lambda F_1(z,\lambda)\Big]\Big\}\lambda^2 f^H(\lambda) J_0(\lambda r)\, d\lambda$$

$$\frac{\sigma_{rz}}{D} = \int_0^\infty \Big\{\frac{hF(z,\lambda,s)}{s} - (h - h\lambda z - z\lambda^2)\frac{F(0,\lambda,s)}{s}e^{-\lambda z} -$$

$$\xi\Big[\frac{h+\gamma}{s}e^{-\omega z} - \frac{\gamma - \gamma z\lambda + z\lambda^2}{s}e^{-\lambda z} - \frac{h}{s}e^{-\gamma z} - \frac{\lambda z}{\lambda+\gamma}e^{-\lambda z} +$$

$$\gamma F_1(z,\lambda)\Big]\Big\}\lambda^2 f^H(\lambda) J_1(\lambda r)\, d\lambda$$

$$F_1(z,\lambda) = \frac{e^{-\gamma z} - e^{-\lambda z}}{\lambda^2 - \gamma^2}, \quad \xi = \frac{1}{s(s+\lambda^2-\gamma^2)}, \quad D = 2mGq_0\frac{\delta^2}{k}$$

The transforms (1.10) can be inverted by using tables from (/15/, p.254 and 257). For the exact solution constructed in this manner, the uniform convergence of the integrals should be confirmed, as had been done in the example given below. We note that it is sufficient for the existence of the Hankel transform $f^H(\lambda)$ and its inversion that $f(r)$ is a function of bounded variation, absolutely integrable on the half-line $[0, +\infty[$ (/16/, p.258). This requirement is obviously always satisfied in physical problems.

2. The formulas obtained above for the exact solution are not always convenient for practical calculations because of their complexity and the presence of eliminable discontinuities.

When investigating processes occurring under intensive heating by concentrated energy fluxes, obtaining a simple mode of solution for small values of the heating time is of special interest. The complete asymptotic expansion of the solution as $t \to 0$ can be obtained by the method developed in /17/. We will confine ourselves here to obtaining a simple asymptotic representation of the solution as $t \to 0$, which is the zero-th term of the expansion in the method in /17/.

We note that one of the first papers devoted to constructing an approximate solution for small times was /18/. The method employed in /17/ is used below because it enables us to obtain **an explicit** estimate of the error of the approximate solution being **obtained.**

The method of extracting the principal terms of the asymptotic form of the solution will be demonstrated in the example of σ_{rz}.

Denoting the operator inverse to L_s by L_t^{-1} (t is the argument of the original) and using first the theorem on integration of the original and then the Taylor formula, we find

$$L_t^{-1}\left\{\frac{1}{s(s+\lambda^2-\gamma^2)}\right\} = t - {}^1/_2 t^2(\lambda^2-\gamma^2)\exp[-(\lambda^2-\gamma^2)\tau], \quad 0<\tau<t \tag{2.1}$$

Consequently, the original of the last two components in the formula for σ_{rz} from (1.10) has the form

$$L_t^{-1}\left\{\int_0^\infty\left[\frac{\lambda z e^{-\lambda z}}{s(\lambda+\gamma)(s+\lambda^2-\gamma^2)} - \frac{\gamma F_1(\lambda, z)}{s(s+\lambda^2-\gamma^2)}\right]\lambda^2 f^H(\lambda) J_1(\lambda r)d\lambda\right\} = \sigma_{rz}^{(1)} + \delta_1 \tag{2.2}$$

$$\sigma_{rz}^{(1)} = t\int_0^\infty\left\{\frac{\lambda z e^{-\lambda z}}{\lambda+\gamma} - \gamma F_1(\lambda, z)\right\}\lambda^2 f^H(\lambda) J_1(\lambda r)\,d\lambda \tag{2.3}$$

where by using (2.1) and the known inequality ((9.1.60) in /19/) $|J_1(x)| \leqslant 1/\sqrt{2}$ **we obtain the following estimate for** δ_1:

$$|\delta_1| \leqslant \frac{t^2 e^{\gamma^2 t}}{2\sqrt{2}}\int_0^\infty\left[\frac{\lambda}{e} + \left(1+\frac{1}{e}\right)\gamma\right]|f^H(\lambda)|\lambda^2\,d\lambda \tag{2.4}$$

Analogously

$$|\delta_2| = \left|L_t^{-1}\left\{\int_0^\infty \frac{(\gamma-\gamma\lambda z-\lambda^2 z)\,e^{-\lambda z}+he^{-\gamma z}}{s^2(s+\lambda^2-\gamma^2)} f^H(\lambda)\lambda^2 J_1(\lambda r)\,d\lambda\right\}\right| \leqslant \frac{t^2}{2\sqrt{2}}e^{\gamma^2 t}\int_0^\infty\left(\gamma+\frac{\lambda}{e}+h\right)|f^H(\lambda)|\lambda^2\,d\lambda \tag{2.5}$$

Setting $y = z/(2\sqrt{t}) + \gamma\sqrt{t}$ and using the well-known formula (/15/, p.248), we furthermore have

$$L_t^{-1}\left\{\frac{e^{-z\sqrt{s}}}{s-\gamma^2}\right\} = \frac{e^{\gamma^2 t}}{2}\left\{e^{-\gamma z}\operatorname{erfc}\left(\frac{z}{2\sqrt{t}}-\gamma\sqrt{t}\right) + e^{\gamma z}\operatorname{erfc}\left(\frac{z}{2\sqrt{t}}+\gamma\sqrt{t}\right)\right\} \leqslant$$
$$\frac{e^{\gamma^2 t}}{2}\left\{2+\frac{2}{\sqrt{\pi}}\exp\left(y^2-\frac{z^2}{4t^2}-\gamma^2 t\right)\int_y^\infty e^{-x^2}dx\right\} =$$
$$\frac{e^{\gamma^2 t}}{2}\left\{2+\exp\left(-\frac{z^2}{4t^2}-\gamma^2 t\right)\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2-2uy}\,du\right\} \leqslant e^{\gamma^2 t}+\frac{1}{2}$$

from which we find that

$$|\delta_3| = \left| L_t^{-1}\left\{\int_0^\infty \frac{(h+\gamma)e^{-z\sqrt{s+\lambda^2}}}{s^2(s+\lambda^2-\gamma^2)} f^H(\lambda)\lambda^2 J_1(\lambda r)d\lambda\right\}\right| \leqslant \tag{2.6}$$
$$(h+\gamma)\left(\frac{t^2}{2}e^{\gamma^2 t} + \frac{t^2}{4}\right)\int_0^\infty |f^H(\lambda)|\lambda^2 d\lambda$$

To estimate the original of the first two components in σ_{rz}^* from (1.10) it is required to estimate the quantity $L_t^{-1}\{F(z, \lambda, s)/s\}$. We will first estimate $L_t^{-1}\{F(z, \lambda, s)\}$. We note that $T_1 = -L_t^{-1}\{F(z, 0, s)\}$ is the solution of the one-dimensional boundary value problem of heat conduction obtained from (1.2) for $f(r) = 1$; $\Delta = \partial^2/\partial z^2$. If $h = \gamma = 0$, then the solution of this problem is $T = T_0 = t$; otherwise $T = T_1 = -L_t^{-1}\{F(z, 0, s)\}$.

Obviously $T_1 \leqslant T_0$ and consequently by taking account of the estimate obtained from (1.3) by using the displacement theorem

$$|L_t^{-1}\{F(z, \lambda, s)\}| \leqslant |L_t^{-1}\{F(z, 0, s)\}|$$

we find by using the theorem on integration of the original

$$\left|L_t^{-1}\left\{\frac{F(z, \lambda, s)}{s}\right\}\right| \leqslant \frac{t^2}{2} \tag{2.7}$$

$$|\delta_4| = \left|L_t^{-1}\left\{\int_0^\infty \left[\frac{hF(z, \lambda, s)}{s} + (\lambda^2 z + h\lambda z - h)\frac{F(0, \lambda, s)}{s}e^{-\lambda z}\right] \times\right.\right.$$
$$\left.\left.\lambda^2 f^H(\lambda) J_1(\lambda r)d\lambda\right\}\right| \leqslant \frac{t^2}{2\sqrt{2}}\int_0^\infty\left(2h + \frac{\lambda}{e}\right)|f^H(\lambda)|\lambda^2 d\lambda$$

It follows from (2.2)-(2.7) that $\sigma_{rz}^{(1)}$ is an asymptotic representation of σ_{rz} as $t \to 0$, where an estimate of the error when σ_{rz} is replaced by $\sigma_{rz}^{(1)}$ does not exceed $|\delta_1| + |\delta_2| + |\delta_3| + |\delta_4|$. Performing analogous calculations for the remaining stresses, we finally obtain

$$\frac{\sigma_{rr}}{D} = -T + t\int_0^\infty \left\{\frac{2-\lambda z}{\lambda+\gamma}e^{-\lambda z} + \lambda F_1(\lambda, z)\right\}\lambda^2 f^H(\lambda) J_0(\lambda r)d\lambda - \tag{2.8}$$
$$t\int_0^\infty\left\{\frac{2-2\nu-\lambda z}{\lambda+\gamma}e^{-\lambda z} + \lambda F_1(\lambda, z)\right\}\lambda^2 f^H(\lambda)\frac{J_1(\lambda r)}{\lambda r}d\lambda + \delta_{rr}$$
$$\frac{\sigma_{\varphi\varphi}}{D} = -T + 2\nu t\int_0^\infty \frac{e^{-\lambda z}}{\lambda+\gamma}\lambda^2 f^H(\lambda) J_0(\lambda r)d\lambda +$$
$$t\int_0^\infty\left\{\frac{2-2\nu-\lambda z}{\lambda+\gamma}e^{-\lambda z} + \lambda F_1(\lambda, z)\right\}\lambda^2 f^H(\lambda)\frac{J_1(\lambda r)}{\lambda r}d\lambda + \delta_{\varphi\varphi}$$
$$\frac{\sigma_{zz}}{D} = t\int_0^\infty\left\{\frac{\lambda z}{\lambda+\gamma}e^{-\lambda z} - \lambda F_1(\lambda, z)\right\}\lambda^2 f^H(\lambda) J_0(\lambda r)d\lambda + \delta_{zz}$$
$$\frac{\sigma_{rz}}{D} = t\int_0^\infty\left\{\frac{\lambda z}{\lambda+\gamma}e^{-\lambda z} - \gamma F_1(\lambda, z)\right\}\lambda^2 f^H(\lambda) J_1(\lambda r)d\lambda + \delta_{rz}$$

Relationships (2.8) are exact expressions for the stresses. The approximate expressions (asymptotically exact as $t \to 0$) are obtained by discarding $\delta_{rr}, \delta_{\varphi\varphi}, \delta_{zz}, \delta_{rz}$ in (2.8) and replacing the temperature in (2.8) by its asymptotic representations. We have

$$|\delta_{rr}| \leqslant t^2\frac{6+\sqrt{2}}{4}\int_0^\infty\{\lambda + h + 2(\lambda+\gamma)e^{\gamma^2 t}\}\lambda^2|f^H(\lambda)|d\lambda \tag{2.9}$$
$$|\delta_{\varphi\varphi}| \leqslant t^2\int_0^\infty\left\{\left(\frac{1}{e} + \frac{1+3\sqrt{2}}{2\sqrt{2}}\right)(h+\lambda) +\right.$$
$$\left.\frac{1+5\sqrt{2}}{2\sqrt{2}}(\lambda+\gamma)e^{\gamma^2 t}\right\}\lambda^2|f^H(\lambda)|d\lambda$$
$$|\delta_{zz}| \leqslant \frac{t^2}{2}\int_0^\infty\left\{\lambda\left(2+\frac{1}{e}\right) + \frac{h}{e} +\right.$$
$$\left.\left[\lambda\left(3+\frac{1}{e}\right) + \frac{2\gamma}{e}\right]e^{\gamma^2 t}\right\}\lambda^2|f^H(\lambda)|d\lambda$$

$$|\delta_{rz}| \leqslant \frac{t^2}{4\sqrt{2}} \int_0^\infty \left\{5h + \gamma + \frac{2\lambda}{e} + \left[\left(\frac{2}{e} + 6\right)\gamma + 4h + \frac{4\lambda}{e}\right] e^{\gamma^2 t}\right\} \lambda^2 |f^H(\lambda)| \, d\lambda$$

Extracting the principal term of the asymptotic form of the temperature, we obtain

$$T = te^{-\gamma z} f(r) + \delta_T \tag{2.10}$$

$$|\delta_T| \leqslant (h + \gamma) t^{3/2} e^{\gamma^2 t} \int_0^\infty \left\{\frac{4}{3\sqrt{\pi}} + \frac{\sqrt{t}}{2}(\lambda^2 + \gamma^2)\right\} \lambda |f^H(\lambda)| \, d\lambda$$

As follows from (2.8)-(2.10), σ_{ij} is of the order of smallness of t, $\delta_{ij} = O(t^2)$, $i, j = r, \varphi, z$ $(t \to 0)$, while $T = O(t)$, $\delta_T = O(t^{3/2})$ $(t \to 0)$. This means that for small values of t the main contribution to the error in the calculation of σ_{rr} and $\sigma_{\varphi\varphi}$ can be introduced by the error in calculating the temperature. To "conserve" the order of the error we extract the first term of the asymptotic expansion of $T(r, z, t)$ by using the method in /17/. We have

$$T = T_1(z, t) f(r) + \varepsilon_T \tag{2.11}$$

$$\varepsilon_T = -\int_0^t \tau \frac{\partial T_1}{\partial \tau} d\tau \int_0^\infty \lambda^3 J_0(\lambda r) f^H(\lambda)^{-\lambda^2 \xi} d\lambda, \quad 0 < \xi < \tau$$

from which, by using the inequality $\partial T_1/\partial \tau \geqslant 0$, $T_1 \leqslant T_0$, we obtain

$$|\varepsilon_T| \leqslant t^2 \int_0^\infty \lambda^3 |f^H(\lambda)| \, d\lambda \tag{2.12}$$

i.e., $\varepsilon_T = O(t^2)$ **is an infinitesimal** of the order needed.

The solution (2.11) has an explicit physical meaning. Indeed, it follows from (2.10) and (2.12) that $\varepsilon_T = o(T)$. This means that for small times heat transfer normal to the surface is realized by one-dimensional heat conduction laws to the accuracy of higher order infinitesimals. Radial heat flux can be neglected to the same accuracy. An analogous result was obtained earlier in the plane case /9/ and in the problem about convective heating of a half-space /20/.

In the case $\gamma = 0$ formulas (2.8) take the form

$$\sigma_{rr}/D = -T + t\{B_{10} - zB_{20} + f(r) - [(1 - 2\nu) B_{01} - zB_{11} + A_{01}]/r\} \tag{2.13}$$

$$\sigma_{\varphi\varphi}/D = -T + t\{2\nu B_{10} + [(1 - 2\nu) B_{01} - zB_{11} + A_{01}]/r\}$$

$$\sigma_{zz}/D = t\{zB_{20} + B_{10} - f(r)\}; \quad \sigma_{rz}/D = tzB_{21}$$

$$A_{ij} = \int_0^\infty \lambda^i J_j(\lambda r) f^H(\lambda) \, d\lambda, \quad B_{ij} = \int_0^\infty \lambda^i J_j(\lambda r) e^{-\lambda z} f^H(\lambda) \, d\lambda \tag{2.14}$$

For certain kinds of distribution the integrals of (2.14) are taken in elementary functions. Thus, for a bell-shaped distribution

$$f(r) = (r^2 + 1)^{-3/2} \tag{2.15}$$

the computations are quite simple while the error estimate (2.9) reduces to factorials (/21/, p.324).

Fig.1 shows lines of equal maximal (solid curves) and minimal (dashes) dimensionless principal stresses $\sigma_1/(Dt)$ and $\sigma_3/(Dt)$ for the distribution (2.15). The stresses were computed by means of (2.13) for $\nu = 0.2$, and the temperature by (2.11). The coefficients (2.14) are presented in /17/. As is seen, near the surface $z = 0$ tensile stresses occur that exceed the compressive stresses in absolute value. This result is thereby qualitatively distinct from the convective heat transfer case /17/.

The stress asymptotic and the error estimate for small t in the case of opaque materials (large γ) are determined by relationships (2.1) and (2.2). However, much simpler relationships can be obtained by different means based on the correctness (stability to small temperature changes) of the thermoelasticity boundary value problem. We will now study this case when slightly transparent **materials are heated by a** concentrated energy flux.

3. Let the temperature be distributed according **to the law (2.10) in** the half-space $z \geqslant 0$ (we later neglect the error δ_T by setting $\delta_T = 0$). If the quantity γ is large, then the temperature

$$T = tf(r)\,e^{-\gamma z}\left(1+\frac{b}{\gamma^2}\right)\quad\left(b=\frac{1}{f(r)}\left(\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}\right)\right) \tag{3.1}$$

does not differ too radically from (2.3) for the distribution $f(r)$ for which b is constrained to the whole range of variation of r. Only such distributions are later considered. (Namely the boundedness of the quantity b for the distribution $f(r) = \exp(-r^2)$ enabled us to use a temperature approximation largely analogous to (3.1) in /7/).

In this case the stresses corresponding to the distribution (3.1) do not differ too radically from the stresses corresponding to the distribution (2.3). But as can be confirmed

$$\Phi = tm\gamma^{-2}f(r)\,e^{-\gamma z} \tag{3.2}$$

is a particular solution of the equation $\Delta\Phi = mT$.

Using the same Love function as above, we find expressions for the stresses

$$\begin{aligned}
\sigma_{rr}/D &= t\gamma^{-2}\{([A_{21}/r - \gamma^2 f(r)]\,e^{-\gamma z} + 2\gamma B_{20} - (1+\gamma z)\,B_{30} + \\
&\quad zB_{40} + [-2\gamma(1-\nu)\,B_{11} + (1-2\nu+\gamma z)\,B_{21} - zB_{31}]/r\} \\
\sigma_{\varphi\varphi}/D &= t\gamma^{-2}\{[A_{30} - \gamma^2 f(r) - A_{21}/r]\,e^{-\gamma z} + 2\nu\,(\gamma B_{20} - B_{30}) + \\
&\quad [2\gamma(1-\nu)\,B_{11} - (1-2\nu+\gamma z)\,B_{21} + zB_{31}]/r\} \\
\sigma_{zz}/D &= t\gamma^{-2}\{A_{30}e^{-\gamma z} + (\gamma z - 1)\,B_{30} - zB_{40}\} \\
\sigma_{rz}/D &= t\gamma^{-2}\{\gamma A_{21}e^{-\gamma z} - \gamma B_{21} + \gamma z B_{31} - zB_{41}\}
\end{aligned} \tag{3.3}$$

corresponding to the temperature (3.1). Here the coefficients A_{ij}, B_{ij} are determined as before by (2.14).

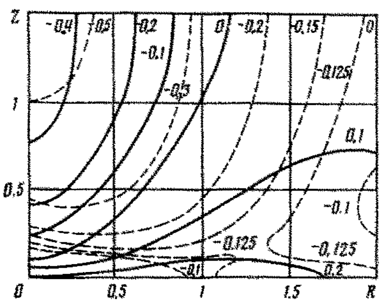

Fig.1

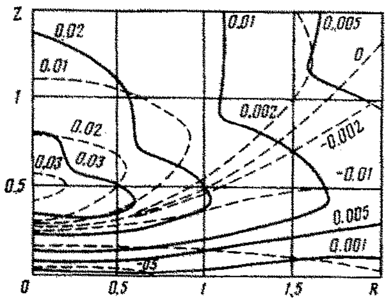

Fig.2

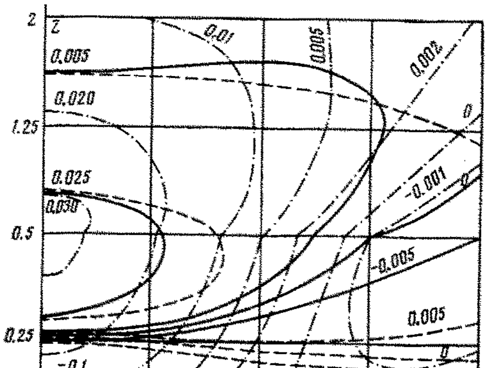

Fig.3

The approximate solution for $f(r) = \exp(-r^2)$ was obtained earlier /7/ in the form of infinite series in A_{ij}. It should noted that the relationships (3.3) have meaning if $\gamma^2 \gg b$. We keep precisely this in mind by saying that the quantity γ is large. More exactly, expressions (3.3) are the asymptotic form of the stresses as $\gamma\to\infty$ and $t\to 0$. In the case of the distribution of heat sources (2.15) the expressions for the stresses (3.2) are elementary functions and have meaning for $\gamma^2 \gg 6$ (which is ordinarily satisfied in practice). Fig.2 shows lines of equal maximal and minimal stresses for the case $\gamma = 10$ and $\nu = 0.2$. The notation agrees with that used in Fig.1. Fig.3 shows lines of equal dimensionless stresses $\sigma_{rr}/(Dt)$ (the solid curves), $\sigma_{\varphi\varphi}/(Dt)$ (the dashed curves) and $\sigma_{zz}/(Dt)$ (the dash-dot curves). The broken lines in the graphs are explained by the three times jumplike change in scale along the z axis (at the point $z = 0.5$) As is seen, the stress distribution is analogous in qualitative respects to the case of convective heat transfer /17/, which is explained in a natural manner by the qualitative similarity of the temperature fields. This is physically evident and follows strictly from the results in /22/.

The error estimate for the approximate solution (3.3) can be obtained by comparison with the approximate solution (2.8) for which there is an explicit error estimate (2.9). When evaluating the integrals in (2.8) the eliminable singularities are extracted in an ε-neighbourhood of γ over which the integral is easily estimated, say, by a Taylor series expansion of the integrand.

4. It should also be noted that another approach can also be used in the case of large γ, according to which heating of the half-space by a concentrated energy flux can be described by a homogeneous heat conduction equation (without taking account of the heat sources) but with heat conduction boundary conditions of the second kind. In this case assigning the heat flux through the half-space surface is substantially equivalent to assigning the heat sources strictly on the half-space surface in the approach utilized above. Details of the temperature and stress field distribution in a surface layer of thickness of the order of $1/\gamma$ remain unknown and we must limit ourselves to large depths at which both approaches yield close results. Consequently, if there is a need to consider moderate depths $(z \leqslant 1/\gamma)$, then it is necessary to take account, somehow explicitly, of the interaction between the radiation and the substance. The simplest method for taking account of both the physical and mathematical viewpoints is to introduce heat sources having a Bouger distribution.

Deviations from the Bouger distribution that occur near the half-space surface should be taken into account in cases when the temperature and stress must be determined in a domain where these deviations are substantial, since these are not felt at great depths. But even then the Bouger law is often used successfully. Thus, in the case of the heating of a half-space by an electron beam the maximum q_m of the internal heat source density lies at a certain depth z_m from the surface /23/ and the distribution function of the internal heat source density is naturally approximated by the expression

$$q(z) = (q_0 + q_1)\, e^{-\gamma z} - q_1 e^{-\beta z} \qquad (4.1)$$
$$q_0 = q(0), \quad q_1 > 0, \quad \gamma < \beta$$

If q_0, q_m, z_m, γ are known, then

$$q_1 = \frac{q_m - q_0 e^{-\gamma z_m}}{e^{-\gamma z_m} - e^{-\beta z_m}}$$

while the quantity β is determined from the equation

$$\frac{\beta}{\gamma}\left(\frac{q_m}{q_0} e^{\gamma z_m} - 1\right) + 1 = \frac{q_m}{q_0} e^{z_m(\beta-\gamma)}$$

which has a unique solution for $\gamma < \beta$.

Therefore the approximation (4.1) results in a linear combination of the solutions obtained above with the absorption coefficients γ and β in this case.

The late V.F. Stal'gorov offered considerable assistance in constructing the graphs.

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ON SOME SPECIAL LAWS OF NON-LINEAR FILTRATION*

G.A. DOMBROVSKII

New laws of the non-linear filtration of an incompressible fluid are proposed (including laws of filtration with a limiting gradient /1/) which enable one, when solving planar, stationary problems, to make use of the apparatus of the theory of functions of a complex variable. Some well-known special cases are considered.

1. The planar stationary filtration of an incompressible fluid is considered. Let $z = x + iy$ be the plane of flow, v be the modulus of the filtration velocity vector, θ be the angle of inclination of the filtration velocity vector to the x-axis, ψ be the stream function, $\varphi = -H + \text{const}$, where H is the head, and let $\Phi(v)$ be a function which characterizes the filtration law /2/. By adopting v and θ as the independent variables, we shall have a system of equations

$$\frac{\partial\varphi}{\partial\theta} = \frac{\Phi^2(v)}{v\Phi'(v)}\frac{\partial\psi}{\partial v}, \quad \frac{\partial\varphi}{\partial v} = -\frac{\Phi(v)}{v^2}\frac{\partial\psi}{\partial\theta}$$

for the functions $\varphi(v, \theta), \psi(v, \theta)$ which can be obtained, for example, from the condition of the integrability of the right-hand side of the differential relationship

$$dz = e^{i\theta}\left[\frac{d\varphi}{\Phi(v)} + \frac{i\,d\psi}{v}\right]$$

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